Soliton Solutions and Stability Analysis for the Akbota Equation

Muhammad Moneeb Tariq [1], Muhammad Bilal Riaz[2,3],Muhammad Aziz-ur-Rehman[1]

[1]Department of Mathematics, University of Management and Technology, Lahore, Pakistan
[2]IT4Innovations, VSB – Technical University of Ostrava, Ostrava, Czech Republic;,
muhammad.bilal.riaz@vsb.cz (M.B.R)
[3]Department of Computer Science and Mathematics, Lebanese American University, Byblos, Lebanon

Abstract

This paper focuses on obtaining exact solutions of nonlinear Akbota equation through the application of the modified Khater method and Sardar sub-equation method. Renowned as one of the latest and precise analytical schemes for nonlinear evolution equations, this method has proven its efficacy by generating diverse solutions for the model under consideration. The governing equation undergoes transformation into an ordinary differential equation through a well-suited wave transformation. This analytical simplification paves the way for the derivation of trigonometric, hyperbolic, and rational solutions through the proposed methods. To illuminate the physical behavior of the model, the study presents graphical plots the selected solutions of Khater and Sardar sub-equation method. This visual representation, achieved by selecting appropriate values for arbitrary parameters, enhances the understanding of the system's dynamics. All calculations in this study are meticulously conducted using the Mathematica and Maple software, ensuring accuracy and reliability in the analysis of the obtained solution. Furthermore we investigate the sensitivity analysis of the dynamical system.

Keywords: Akbota equation, Solitary wave analysis, Exact solutions, The modifed Khater method, Sardar sub-equation method, Bifurcation analysis, Sensitivity analysis

1 Introduction

In several scientific domains, including hydrodynamics, nuclear science, material science, plasma physics, and related disciplines, nonlinear partial differential equations (NPDEs) are employed to investigate resonance physical phenomena of diverse applications and dynamical approaches. There is no one-size-fits-all strategy for solving NPDEs, despite the fact that numerous efficient approaches have been developed. Partial differential equations (PDEs) have therefore always been a challenging mathematical problem to solve.Researchers have found many ways to solve NPDEs, numerous including the Hirota bilinear approach [1] the Hirota direct method [2], Improved Kudryashov [3],H.perturbation technique [4],JE function scheme [5],exp $\phi(\xi)$-expansion method [6], the MSE method [7], the extended tanh technique [8],the EMSE technique [9],the bifurcation technique [10],the unified approach [11] , the Wronskian approach [12],extension ERF method [13],LSP method [11],Wang's$\ddot{B}$acklund transformation-based method [12] etc. Besides these some fractional technique, such as truncated M-fractional derivative [14] local fractional derivative [15],fractal generalized variational structure [16], conformable derivative [17],Caputo-Fabrizio derivative [18] and so on are established by various researchers.

A nonlinear wave packet that maintains its original shape while propagating at a constant speed is called a soliton, or solitary wave. Solitons are self-reinforcing and surprisingly stable, even after interacting or colliding with other solitons, in contrast to regular waves that disperse or evaporate. They are a significant subject in both mathematics and physics because of this special quality. John Scott Russell made the first observation of the phenomena of solitons in 1834 when he reported a "great wave of translation" that was moving up a Scottish canal without altering form [19]. Despite being unappreciated at first, his discovery served as the basis for subsequent theoretical advancements. Boussinesq and Lord Rayleigh contributed mathematical insights into solitary waves in the late 19th century, while Korteweg and de Vries developed the famous KdV equation in 1895, creating the first rigorous mathematical model of soliton solutions [20].Significant progress in theoretical physics and applied mathematics has resulted from their research. Recent studies have demonstrated the use of solitons in a wide range of domains, including condensed matter systems, optical fiber communication, plasma physics, shallow water wave dynamics, and even biological models [21].

In various branches of physics and chemistry, the real-valued Ginzburg–Landau equation also known as the Allen–Cahn equation or the nonlinear heat equation—occurs. In the context of pattern development in the case of convection in binary mixes close to instability, it was initially created as a long-wave amplitude equation. It has been extensively researched as a deterministic model [22]. In recent years, physics, biology, chemistry, climatic dynamics, geophysics, and other fields have generally acknowledged the need to take random effects into account when modeling, forecasting, evaluating, and simulating complex processes [23]. Stochastic differential equations are those that take time-dependent random fluctuations into account. This article examines the following stochastic real-valued Ginzburg-Landau equation that is forced in the Itô sense by multiplicative noise:

$$u_{xx} + u + u_3 + \sigma u \beta_t \quad (1)$$

where $\beta(t)$ is the standard Wiener process and $\beta_t = \frac{d\beta}{dt}$. Here in this study, we consider only the noise is a constant in the space. Let us first define the Brownian motion $\beta(t)$. The stochastic process, which satisfies: (i)$\beta(t)$,$t \geq 0$ are continuous functions of t (ii) For $s < t$,$\beta(t)$ $\beta(s)$is independent of the past, this is also referred to as independence of increments (iii)$\beta(t)$ $\beta(s)$has a normal distribution with mean θ and variance t s, is called a Brownian motion. The existence and uniqueness of Eq. (1) were widely studied in literature see for instance [24]. In the other hand, there are different approaches for seeking solutions of real-valued Ginzburg–Landau Eq. (1). Some of those approaches are the perturbation method [25], the Green function [26]. Our goal in this paper is to derive the exact stochastic solutions of stochastic real-valued Ginzburg–Landau Eq. (1) forced by a one- dimensional multiplicative white noise in the Itô sense by using the Riccati method.

To explore the nonlinear behaviors described by the Ginzburg–Landau equation, it is essential to recognize that the framework of complex systems offers a foundation for examining dynamical systems involving numerous interacting variables.One way to describe the mathematical discipline of complex systems theory is as the study of dynamical systems (DS) that are characterized by a high number of variables. Enhancing the fundamentals of DS theory, which primarily addresses systems with a limited number of variables, is the main goal of this field. Numerous fields, such as robotics, network engineering, telecommunication, and scientific research, have made substantial use of these technologies. Nonlinear partial differential equations, such as the Ginzburg–Landau equation, primarily describe complicated physical phenomena in this regard. The dynamical behaviors of this model, including an investigation of its equilibrium and perturbed states, are the main emphasis of this paper.We examine several topics that can arise in the given DS, such as sensitivity, bifurcation analysis, chaotic responses, and multistability analysis which together provide a comprehensive picture of the rich dynamics inherent in the system.

Numerous analytical techniques have been used to find soliton solutions for different NLPDEs. Analytical solutions are useful because they allow for more accurate modeling of system behaviors that numerical techniques cannot replicate, as well as a deeper understanding of fundamental physical mechanisms. Because NLPDEs necessitate ongoing research as a foundational topic, researchers continue to place a high priority on the current investigation of analytical solutions. To address the difficulties in solving NLPDEs, researchers have developed a number of analytical and numerical methods.

When using these analytical techniques, choosing an appropriate strategy is essential. Among those approaches, the generalizing Riccati equation mapping method [27] is a reliable and valid process for developing new, precise soliton solutions. The recommended approach avoids tedious and repetitive computations. The analysis of complex biological and physical processes is made easier by the generalizing Riccati equation mapping technique. Consequently, the soliton solutions for the proposed model are obtained in this study by applying the generalizing Riccati equation mapping technique.

Bifurcation analysis is currently crucial for understanding how small changes in system characteristics can have a big impact on how dynamical systems behave. This mathematical framework helps determine critical moments when a system changes states, which is essential for forecasting. and managing intricate behaviors in several domains [28,29] . In engineering, bifurcation analysis is used to predict and prevent structural issues. Engineers can design stronger buildings and avert catastrophic outcomes by understanding how systems respond to changes in parameters. In fluid dynamics, bifurcation analysis clarifies the transition from laminar to turbulent flow and provides details regarding the formation of vortices and patterns [30].In biology, this research looks at disease transmission, population dynamics, and neural activity, showing how small changes can have a big impact on patterns of behavior. In economics, bifurcation analysis helps simulate market dynamics and predict how slight changes in economic expansions or contractions can be caused by factors like interest rates. Understanding chaotic systems requires an understanding of bifurcation analysis. It helps predict chaotic behavior in both natural and artificial systems and pinpoints the paths leading to chaos. Bifurcation analysis is a powerful tool for managing and analyzing complex systems, providing crucial information to researchers and practitioners alike. This study aims to apply bifurcation theory to Ginzburg–Landau equation and to create the traveling wave. We obtain the phase portrait and Hamiltonian function under various parametric conditions.

A nonlinear dynamical system (DS) is said to behave consistently when it deviates from fixed or repeating trajectories throughout time, a phenomenon known as chaos [31]. It occurs within perfectly deterministic and simple DS, while it may appear to be a random fluctuation. The capacity to produce precise long-term predictions and results is hampered by chaotic systems' extreme sensitivity to initial conditions(ICs), despite their deterministic operation. Continuous-time systems with at least three independent dynamical variables are susceptible to chaos, as long as they are also nonlinear and exhibit characteristics like Lyapunov exponents and Poincaré maps. In continuous-time systems, several requirements must be met in order to identify chaos. When a system is sensitive to its ICs, it is considered chaotic. Fundamentally, the full definition of chaos is that a chaotic system behaves in a way that seems random and unpredictable but is actually governed by deterministic laws [32]. The complicated propagation of nonlinear waves is mostly described by chaos in the context of nonlinear partial differential equations (NLPDEs). In order to comprehend bifurcation patterns, multistability, and the transition between coherent soliton states and irregular waveforms, it is particularly crucial to study chaos in NLPDEs, such as the Ginzburg–Landau equation. The underlying mechanisms of nonlinear wave propagation in physical systems can only be revealed by getting exact or approximate analytical solutions, notwithstanding the mathematical complexity of these equations.

A technique for assessing the impact of different hypotheses or analyses on the pre-formulated research questions is sensitivity analysis [33, 34]. Stated differently, the goal of a sensitivity study is to assess the reliability and validity of the main analytical or methodological approach. When there are a variety of plausible and divergent assumptions, sensitivity studies are most instructive [35]. A single or a collection of mathematical models that are encoded using computer software and replicate the operation of a real-world system of interest are used in various applications. Such mathematical models can be mechanistic (also known as process-based) [36], which solves a series of differential equations or other mathematical equations governing the (potentially) spatiotemporal behaviors of the underlying processes, or data-driven (also known as statistical), which directly maps inputs to outputs [37].

2 Proposed Method

2.1 Discruption of Riccati method

Consider the NLPDE of the form

$$F(u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, ...) = 0 \tag{2}$$

where $u = u(x, t)$ is unknown function.
Step 1: Select the transformation

$$u(x, t) = \phi(\mu)e^{\sigma\beta(t)-\sigma^2 t}, \mu = c(x - \lambda t), \tag{3}$$

which convert the Eq. (1) into Ordinary Differential equation (ODE).

$$G(\mu, \mu', \mu'', \mu''', ...) = 0 \tag{4}$$

Step 2 : Suppose the General solution of the ODE is

$$\phi(\mu) = \beta_0 + \sum_{j=1}^{N} \beta_j Q^j(\mu), \beta_j \neq 0. \tag{5}$$

where β_j (1 ≤ j ≤ n) are the arbitrary constants and $Q^j(\zeta)$ is the solution of the equation:

$$Q^{'}(\mu) = \gamma_2 Q^2(\mu) + \gamma_1 Q(\mu) + \gamma_0 \tag{6}$$

Where γ_2 , γ_1 and γ_0 are real constants and (3) yields the sets of solutions as:
Step 3 : To determine the value of n we use balancing method among the highest derivative and nonlinear term.
Step 4 : substitute Eqs. (6) with (5) in Eq. (4) and we get the algebric equation with similar power of $\phi(\mu)$ with the help of mathematica and obtained the value of β_0,β_1 and α_1.

3 Solution of Riccati method

In this section, we present the Riccati method as a tool for constructing a new wave pattern in a hypothetical model.

Select the transformation

$$u(x, t) = \phi(\mu)e^{\sigma\beta(t)-\sigma^2 t}, \mu = c(x - \lambda t), \tag{7}$$

After applying the selected transformation on Eq.(1), we get

$$c^2\phi^{''} + c\lambda\phi' - \phi^3 + (1 + \frac{1\sigma^2}{2})\phi \tag{8}$$

We focus on obtaining the solitary wave solutions for Eq. (1). The process involves employing the balancing method between higher order derivative and nonlinear term ϕ'' and ϕ^3 to determine the value from Eq. (13). Subsequently, we obtained the value of $N = 1$, inserted into Eq. (5), allowing for the derivation of further insights we get:

$$\phi(\mu) = \beta_0 \ + \beta_1 Q(\mu). \tag{9}$$

By incorporating the expressions from Eq. (14) with (5) and (6) into Eq. (13), and subsequently relating all terms with similar powers of $\phi(\mu)$, to establish a systematic and comprehensive representation of the equation.

$$\begin{aligned}
&Q^0(\mu) : (c\lambda\beta_1\alpha_0 - \beta_0^3 + (1 + \frac{1\sigma^2}{2})) = 0,\\
&Q^1(\mu) : (c^2\beta_1(2\alpha_0\alpha_2 + \alpha_1^2) + c\lambda\beta_1\alpha_1 - 3\beta_0^2\beta_1 + (1 + \frac{1\sigma^2}{2})\beta_1) = 0,\\
&Q^2(\mu) : (3c^2\alpha_1\alpha_2\beta_1 + c\lambda\alpha_2\beta_1 - 3\beta_0\beta_1^2) = 0,\\
&Q^3(\mu) : 2c^2\alpha_2^2\beta_1 - \beta_1^3 = 0,
\end{aligned}$$

The solution sets have been obtained by systematically solving the complex system of algebraic equations, utilizing the sophisticated assistance of powerful computational tools such as Maple or Mathematica. We acquire the valvue of β_0, β_1 and α.

$$\lambda_0 = 0, \quad \alpha_1 = \sqrt{\frac{2c^2\alpha_0\alpha_2 + \frac{1\sigma^2}{2}}{c}}, \quad \beta_0 = \sqrt{2c^2\alpha_0\alpha_2 + \frac{1\sigma^2}{2}} + 1, \; \beta_1 = \sqrt{2}\alpha_2 c$$

Case 1: When $\Delta = \alpha_1^2 - 4\alpha_0\alpha_2 > 0$, $\alpha_1\alpha_2 \neq 0$,(or$\alpha_0\alpha_2 \neq\neq 0$) and non-zero real constants F_1 and F_2,the hyperbolic solutions are:

$$\begin{aligned}
Q_1(\mu) &= \sqrt{2}\alpha_2 c(-\frac{1}{2}\frac{\sqrt{\Delta}tanh(\frac{1}{2}\sqrt{\Delta}(\mu+\zeta_0))}{\alpha_2} - \frac{1}{2}\frac{\alpha_1}{\alpha_2}) + \frac{1}{2}\sqrt{8c^2\alpha_0\alpha_2 + 2\sigma^2 + 4},\\
Q_2(\mu) &= \sqrt{2}\alpha_2 c(-\frac{1}{2}\frac{\sqrt{\Delta}coth(\frac{1}{2}\sqrt{\Delta}(\mu+\zeta_0))}{\alpha_2} - \frac{1}{2}\frac{\alpha_1}{\alpha_2}) + \frac{1}{2}\sqrt{8c^2\alpha_0\alpha_2 + 2\sigma^2 + 4},\\
Q_3^{\pm}(\mu) &= \sqrt{2}\alpha_2 c(-\frac{1}{2}\frac{\sqrt{\Delta}(tanh(\sqrt{\Delta}(\mu+\zeta_0)) \pm \iota sech(\sqrt{\Delta}(\mu+\zeta_0)))}{\alpha_2} - \frac{1}{2}\frac{\alpha_1}{\alpha_2}) + \frac{1}{2}\sqrt{8c^2\alpha_0\alpha_2 + 2\sigma^2 + 4},\\
Q_4(\mu) &= \sqrt{2}\alpha_2 c(-\frac{1}{4}\frac{\sqrt{\Delta}(tanh(\frac{1}{4}\sqrt{\Delta}(\mu+\zeta_0)) + coth(\frac{1}{4}\sqrt{\Delta}(\mu+\zeta_0)))}{\alpha_2} - \frac{1}{2}\frac{\alpha_1}{\alpha_2}) + \frac{1}{2}\sqrt{8c^2\alpha_0\alpha_2 + 2\sigma^2 + 4},\\
Q_5^{\pm}(\mu) &= \frac{1}{2}\frac{\pm\sqrt{\Delta(F_1^2+F_2^2)} - F_1\sqrt{\Delta}cosh(\sqrt{\Delta}(\mu+\zeta_0))}{\alpha_2(F_1 sinh(\sqrt{\Delta}(\mu+\zeta_0)+F_2))} - -\frac{1}{2}\frac{\alpha_1}{\alpha_2}) + \frac{1}{2}\sqrt{8c^2\alpha_0\alpha_2 + 2\sigma^2 + 4},\\
Q_6^{\pm}(\mu) &= \sqrt{2}\alpha_2 c\frac{2rcosh(\sqrt{\Delta}(\mu+\zeta_0))}{\sqrt{\Delta}sinh(\sqrt{\Delta}(\mu+\zeta_0)) - pcosh(\sqrt{\Delta}(\mu+\zeta_0)) \pm \iota\sqrt{\Delta}} + \frac{1}{2}\sqrt{8c^2\alpha_0\alpha_2 + 2\sigma^2 + 4},\\
Q_7^{\pm}(\mu) &= \sqrt{2}\alpha_2 c\frac{2rsinh(\sqrt{\Delta}(\mu+\zeta_0))}{\sqrt{\Delta}cosh(\sqrt{\Delta}(\mu+\zeta_0)) - psinh(\sqrt{\Delta}(\mu+\zeta_0)) \pm \sqrt{\Delta}} + \frac{1}{2}\sqrt{8c^2\alpha_0\alpha_2 + 2\sigma^2 + 4},
\end{aligned} \tag{10}$$

Case 2: For $\Delta = \alpha_1^2 - 4\alpha_0\alpha_2 < 0$, $\alpha_1\alpha_2 \neq 0$,(or$\alpha_0\alpha_2 \neq\neq 0$),and $F_1^2 - F_2^2 > 0$,the trigonometric solutions are:

$$\begin{aligned}
Q_8(\mu) &= \sqrt{2}\alpha_2 c(\frac{1}{2}\frac{\sqrt{-\Delta}tan(\frac{1}{2}\sqrt{-\Delta}(\mu+\zeta_0))}{\alpha_2} - \frac{1}{2}\frac{\alpha_1}{\alpha_2}) + \frac{1}{2}\sqrt{8c^2\alpha_0\alpha_2 + 2\sigma^2 + 4},\\
Q_9(\mu) &= \sqrt{2}\alpha_2 c(-\frac{1}{2}\frac{\sqrt{-\Delta}cot(\frac{1}{2}\sqrt{-\Delta}(\mu+\zeta_0))}{\alpha_2} - \frac{1}{2}\frac{\alpha_1}{\alpha_2}) + \frac{1}{2}\sqrt{8c^2\alpha_0\alpha_2 + 2\sigma^2 + 4},\\
Q_{10}(\mu) &= \sqrt{2}\alpha_2 c(\frac{1}{2}\frac{\sqrt{-\Delta}(tan(\sqrt{-\Delta}(\mu+\zeta_0)) \pm sec(\sqrt{-\Delta}(\mu+\zeta_0)))}{\alpha_2} - \frac{1}{2}\frac{\alpha_1}{\alpha_2}) + \frac{1}{2}\sqrt{8c^2\alpha_0\alpha_2 + 2\sigma^2 + 4},\\
Q_{11}^{\pm}(\mu) &= \sqrt{2}\alpha_2 c(\frac{1}{4}\frac{\sqrt{-\Delta}(tan(\frac{1}{4}\sqrt{-\Delta}(\mu+\zeta_0)) - cot(\frac{1}{4}\sqrt{-\Delta}(\mu+\zeta_0)))}{\alpha_2} - \frac{1}{2}\frac{\alpha_1}{\alpha_2}) + \frac{1}{2}\sqrt{8c^2\alpha_0\alpha_2 + 2\sigma^2 + 4},\\
Q_{12}(\mu) &= \frac{1}{2}\frac{\pm\sqrt{-\Delta(F_1^2-F_2^2)} - F_1\sqrt{-\Delta}cos(\sqrt{-\Delta}(\mu+\zeta_0))}{\alpha_2(F_1 sinh(\sqrt{-\Delta}(\mu+\zeta_0)+F_2))} - \frac{1}{2}\frac{\alpha_1}{\alpha_2}) + \frac{1}{2}\sqrt{8c^2\alpha_0\alpha_2 + 2\sigma^2 + 4},\\
Q_{13}^{\pm}(\mu) &= \sqrt{2}\alpha_2 c\frac{2\alpha_0 cos(\sqrt{-\Delta}(\mu+\zeta_0))}{\alpha_1 cos\sqrt{-\Delta}(\mu+\zeta_0) + \sqrt{-\Delta}sin(\sqrt{-\Delta}(\mu+\zeta_0)) \pm \sqrt{\Delta}} + \frac{1}{2}\sqrt{8c^2\alpha_0\alpha_2 + 2\sigma^2 + 4},\\
Q_{14}\pm(\mu) &= \sqrt{2}\alpha_2 c\frac{2\alpha_0 sin(\sqrt{-\Delta}(\mu+\zeta_0))}{\alpha_1 sin\sqrt{-\Delta}(\mu+\zeta_0)) - \sqrt{-\Delta}cos(\sqrt{-\Delta}(\mu+\zeta_0)) \pm \sqrt{-\Delta}} + \frac{1}{2}\sqrt{8c^2\alpha_0\alpha_2 + 2\sigma^2 + 4},
\end{aligned} \tag{11}$$

Case 3: The rational solutions are :

(i) When $\alpha_0 = \alpha_1 = 0$ and $\alpha_2 \neq 0$ then solution is :

$$Q_{15}^{\pm}(\mu) = \sqrt{2}\alpha_2 c \pm \frac{1}{\alpha_2(\mu+\zeta_0)} + \frac{1}{2}\sqrt{2\sigma^2+2}, \tag{12}$$

(ii) For $\alpha_0, \alpha_2 = \frac{\alpha_1^2}{4\alpha_0}$, then the solution is:

$$Q_{16}^{\pm}(\mu) = -\frac{1}{2}\frac{\sqrt{2}c\alpha_0(\alpha_1(\mu+\zeta_0)+2)}{\alpha_0(\mu+\zeta_0)} + \frac{1}{2}\sqrt{2c^2\alpha_1^2 + 2\sigma^2 + 4}, \tag{13}$$

Case 4: The exponential solutions are:

(i) For $\alpha_2 = 0 and \alpha_0 = \nu\gamma_1$, the solution is:

$$Q_{17}(\mu) = \frac{1}{3}\frac{\sqrt{-\frac{2}{\lambda}}\zeta\lambda + \sqrt{-2\zeta^2\lambda}}{\lambda} \tag{14}$$

(ii) For $\alpha_0 = 0 and \alpha_1 \neq 0, \alpha_2 \neq 0$, the solutions are:

$$Q_{18} = \frac{\sqrt{2}c\alpha_1\varrho}{\exp^{-\alpha_1(\mu+\zeta_0)} - \varrho} + \frac{1}{2}\sqrt{2\sigma^2 + 4} \tag{15}$$

3.1 Graphical Representation of Graphs

To illustrate the physical significance of the model, we present density and contour diagrams, as well as three-dimensional and two-dimensional diagrams that display the results that have been obtained. With a strong emphasis on selecting appropriate parameter values, the visual representations are meticulously crafted with the help of Mathematica. A comprehensive understanding of the implications of the diagrams is provided by the diagrams, which offer an insightful presentation.

4 Dynamical analysis

A vital technique for understanding mathematical models is dynamical analysis. Different elements of the equation can be explored thanks to the variety of instruments used in this study. First, the existence and nature of solutions are understood through the use of phase portraiture. These images show the existence of several solutions in different regions of the axis and offer a clear visual representation for analyzing solutions with regard to initial value difficulties. Second, a critical analysis of the model's nature with regard to chaos is conducted. To comprehend the chaotic behavior of the models, two distinct phenomena are taken into consideration. Finally, a multistability analysis is carried out to ascertain the model's stability with regard to starting value issues. This analysis reveals the range and conditions under which the model remains stable.

4.1 Phase Potrait

Regardless of whether the parameters are interdependent or not, bifurcation analysis looks at dynamical systems and observes how the system behaves at different parameter values. A two-dimensional depiction of phase space that displays each variable's current state in respect to the others is called a phase portrait. In the phase portrait, a periodic solution is shown as a closed curve, whereas an equilibrium point denotes a particular, static state. The trajectories of chaotic solutions are shown to be irregular and distinct. To investigate the emerging dynamics system of Ginzburg Landau equation by applying the principles of bifurcation theory. Upon simplification, eq. (13) expressed in the form of a planar dynamical system as follows:

$$\begin{cases} \frac{d\phi}{d\mu} &= Y, \\ \frac{dY}{d\mu} &= \frac{\lambda\phi'}{c} + \gamma_1(\phi(\mu))^3 - \gamma_2\phi(\mu) \end{cases} \tag{16}$$

Where $\gamma_1 = -\frac{1}{c^2}$,$\gamma_2 = \frac{2+\sigma^2}{2c^2}$. This system exhibits Hamiltonian characteristics and possesses the following integral:

$$G(u, y) = \frac{Y^2}{2} - \frac{\lambda}{c}\phi(\mu) - \gamma_1\frac{\phi^3(\mu)}{3} + \gamma_2\frac{\phi^2(\mu)}{2} \tag{17}$$

In the presence of the Hamiltonian constant, we explore the bifurcations of the phase portraits within the parameter space defined by γ_1, γ_2 and γ_3 and γ_3 for eq. (23).

By qualitative analysis, the three equilibrium points for the system of differential equations are determined to be (0, 0), $(D_1, 0)$, and $(D_2, 0)$ along the D-axis, where D_1 and D_2 represent

$$D_1 = (\sqrt{\frac{\gamma_1}{\gamma_2}}), D_2 = (-\sqrt{\frac{\gamma_1}{\gamma_2}}). \tag{18}$$

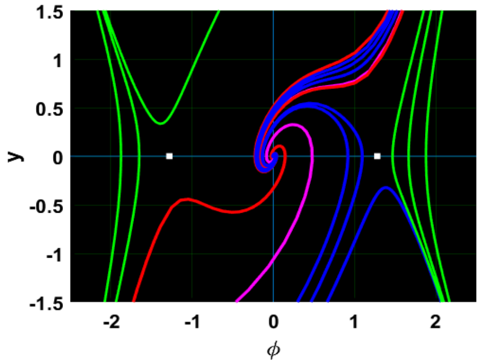
Also, the jacobian of the system are:

$$det(J(d, y)) = \begin{vmatrix} 0 & 1 \\ 3\gamma_1(\phi(\mu))^2 - \gamma_2 & 0 \end{vmatrix} = -3\gamma_1(\phi(\mu))^2 + \gamma_2. \tag{19}$$

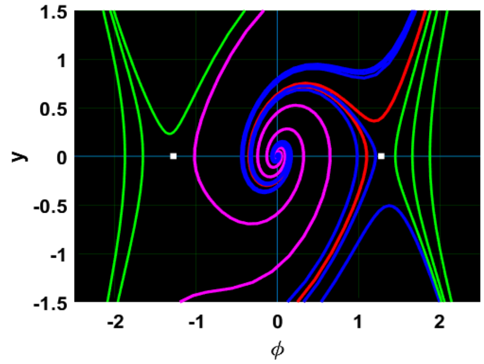
Thus (d, y) is a saddle point for(det $J(d, y)) < 0$, it is a center for $(detJ(d, y)) > 0$ and a cuspidal point for $(detJ(d, y)) = 0$. Various outcomes can be achieved by assigning different values to the involved parameters.

Case 1: Under this case,$\gamma_1 < 0$ and $\gamma_2 > 0$. we get three equilibrium point $M_1 = (0, 0)$, $M_2 = (D_1, 0)$ and $M_3 = (D_2, 0)$. By using $c = 1$,$\sigma = 2$,$\phi = 3$, and obtained the $M_1 = (0, 0)$, $M_2 = (1, 0)$ and $M_3 = (-1, 0)$. Here we see in figure 14 M_1 and M_3 represent the center point and M_2 represent the saddle point.

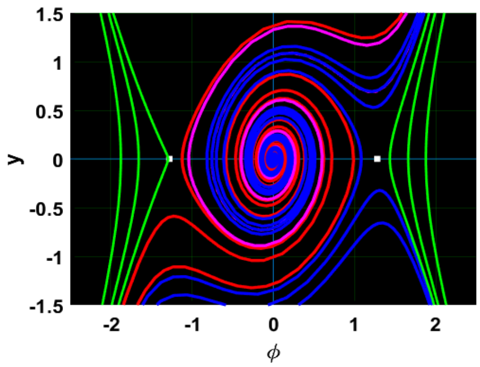
Figure 1: Phase portrait of Case 1 with $\sigma_1 > 0$ and $\sigma_2 < 0$.



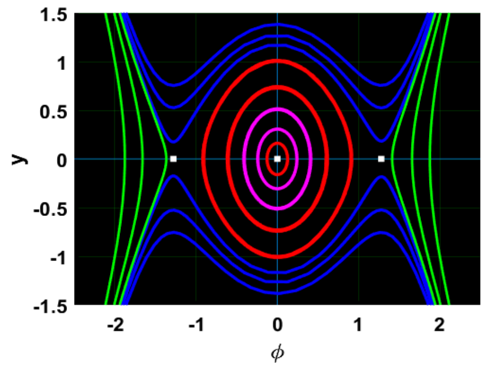
(a) For $\alpha_1 > 0$ and $\alpha_2 > 0$



(b) For $\alpha_1 < 0$ and $\alpha_2 > 0$.



(c) For $\alpha_1 > 0$ and $\alpha_2 < 0$.



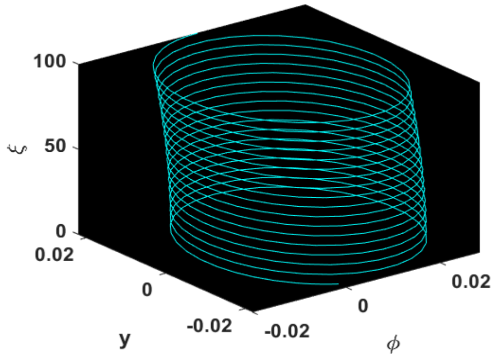
(d) For $\alpha_1 < 0$ and $\alpha_2 < 0$

Figure 2: Phase portrait analysis of system (**??**)

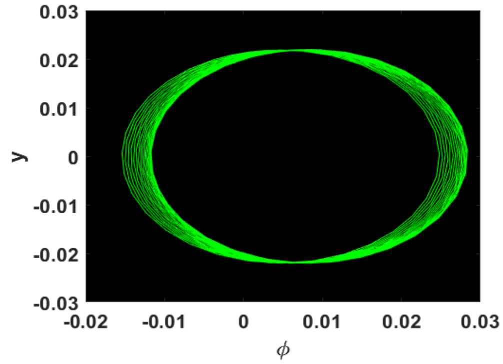
4.2 Chaos analysis

The complicated and erratic behavior present in certain nonlinear systems is referred to as chaos. The goal of chaos analysis is to comprehend and characterize this behavior. The existence of odd attractors fractal patterns in phase space that system paths gravitate toward—is one of the primary indicators of chaos in phase plane analysis. Because of how sensitive these attractors are to beginning conditions, even minor adjustments can have a significant impact on the system's long term behavior.

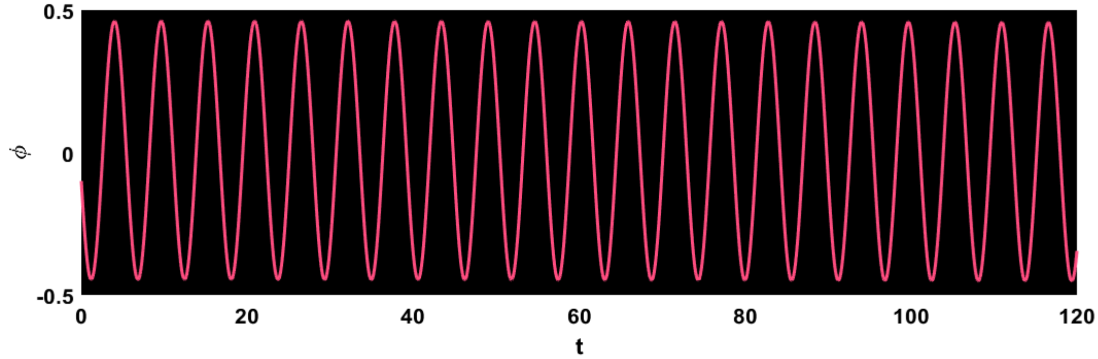
$$\begin{cases} \frac{d\phi}{d\mu} &= Y, \\ \frac{dY}{d\mu} &= \frac{\lambda\phi'}{c} + \gamma_1(\phi(\mu))^3 - \gamma_2\phi(\mu) + \alpha cos(\beta\mu) \end{cases} \tag{20}$$



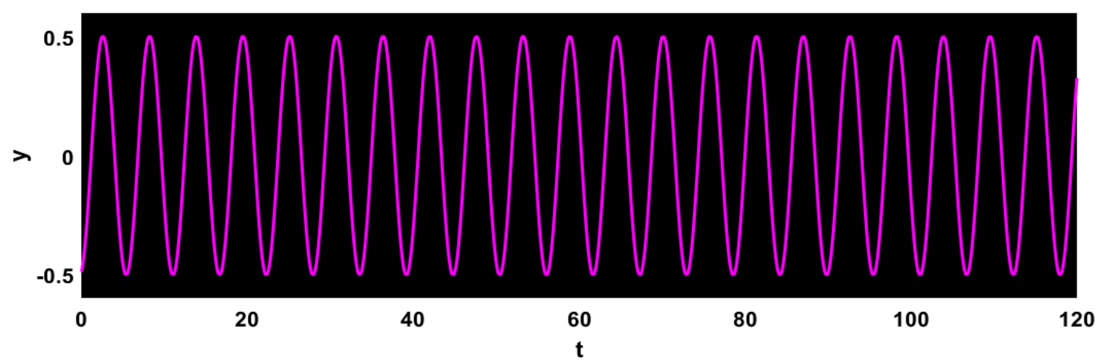
(a)



(b)

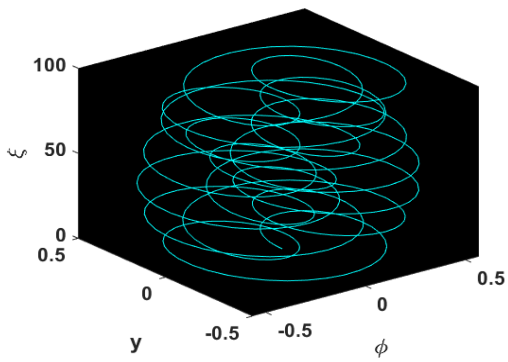


(c)

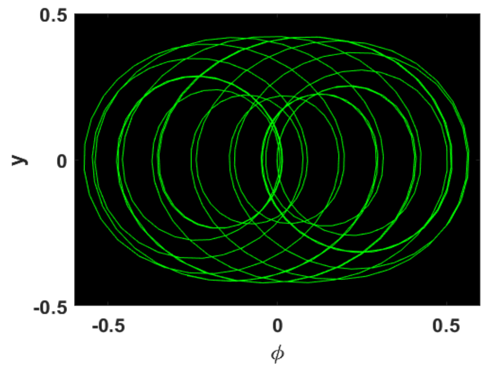


(d)

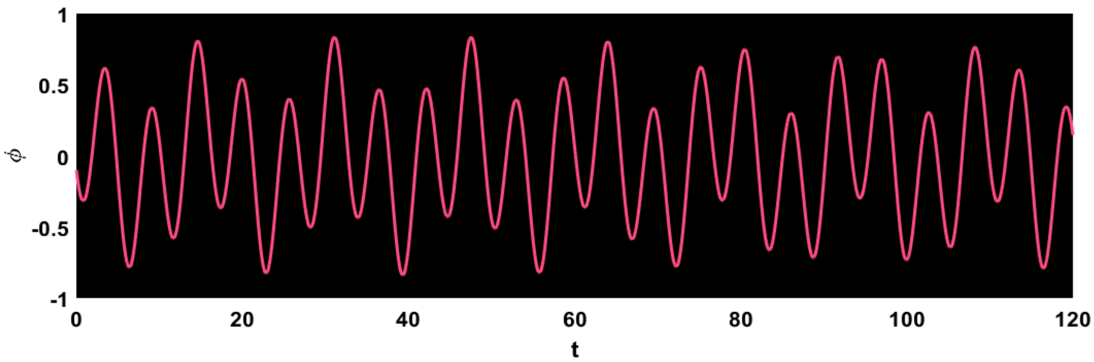
Figure 3



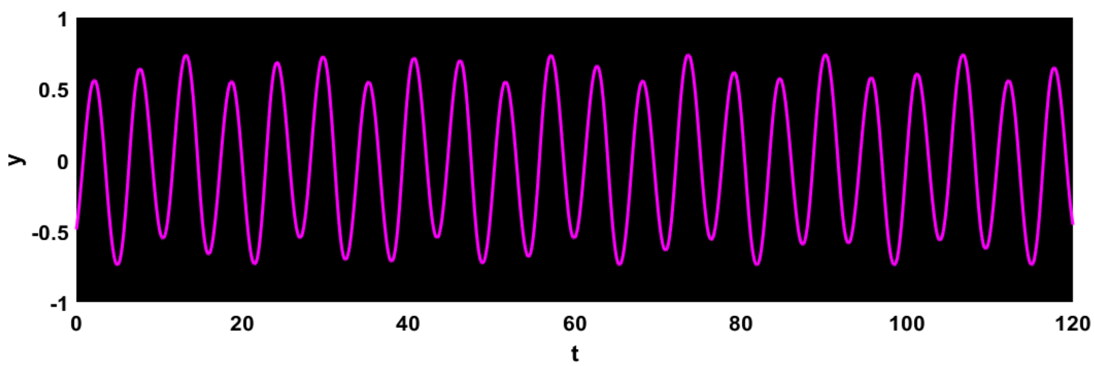
(a)



(b)



(c)



(d)

Figure 4

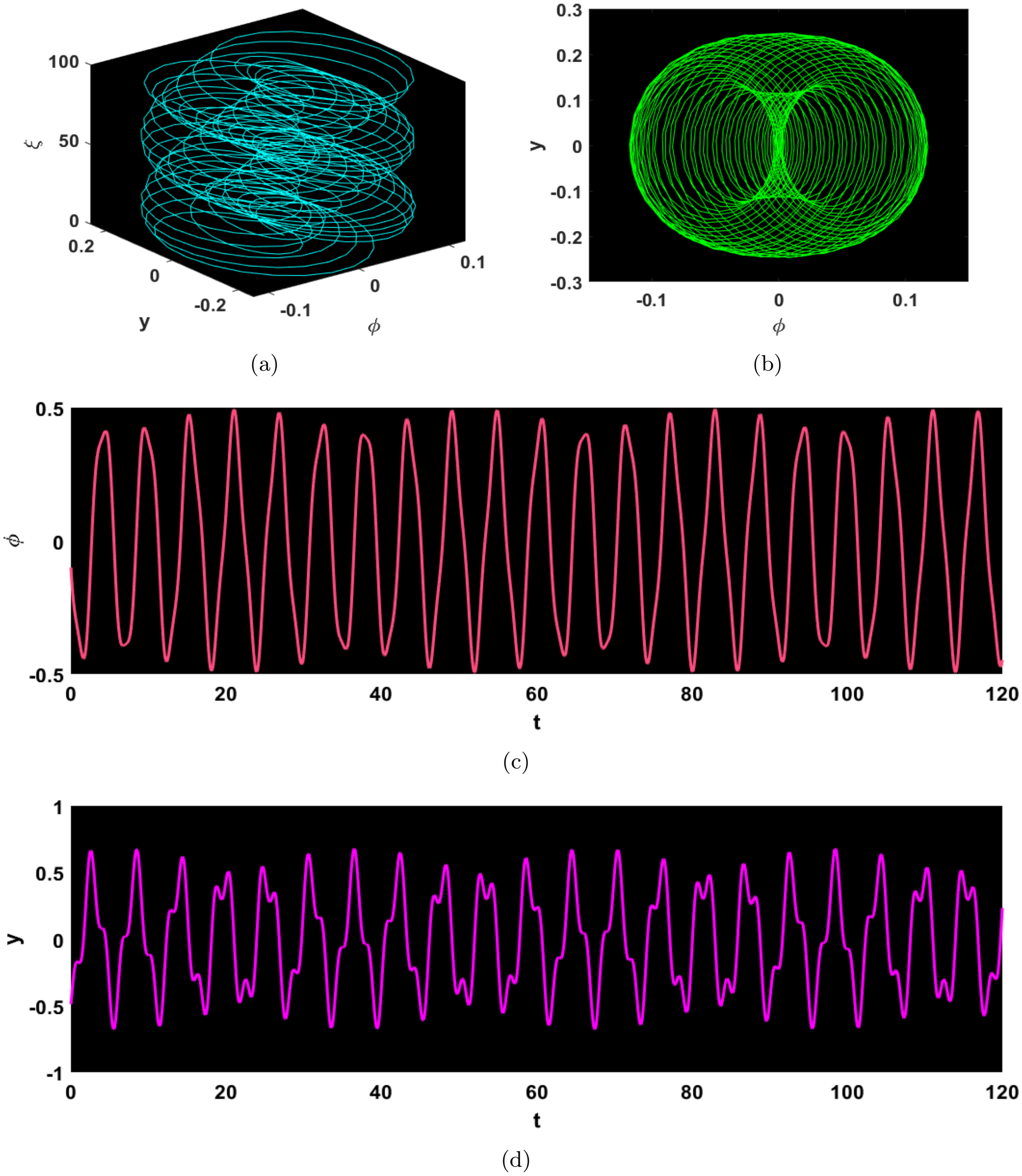
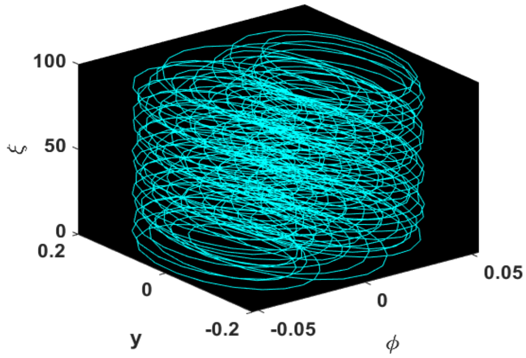
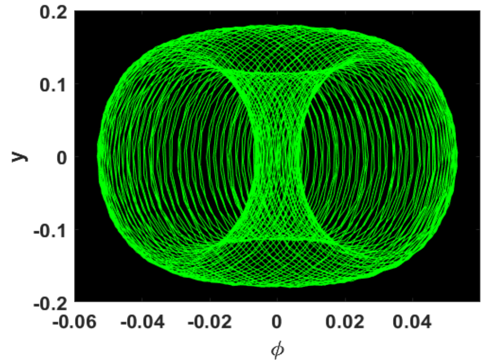

100
ξ
50
0
0.2
0
y
-0.2
-0.1
0
0.1
ϕ
(a)
0.3
0.2
0.1
y
0
-0.1
-0.2
-0.3
-0.1
0
0.1
ϕ
(b)
0.5
ϕ
0
-0.5
0
20
40
60
80
100
120
t
(c)
1
0.5
y
0
-0.5
-1
0
20
40
60
80
100
120
t
(d)

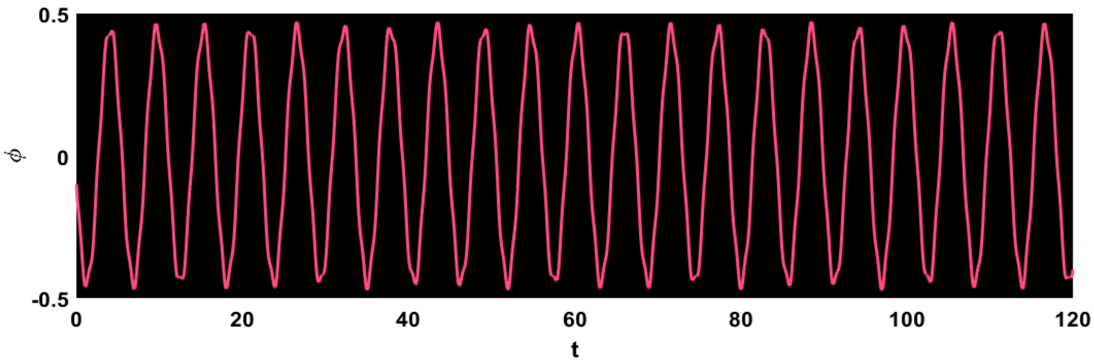
Figure 5



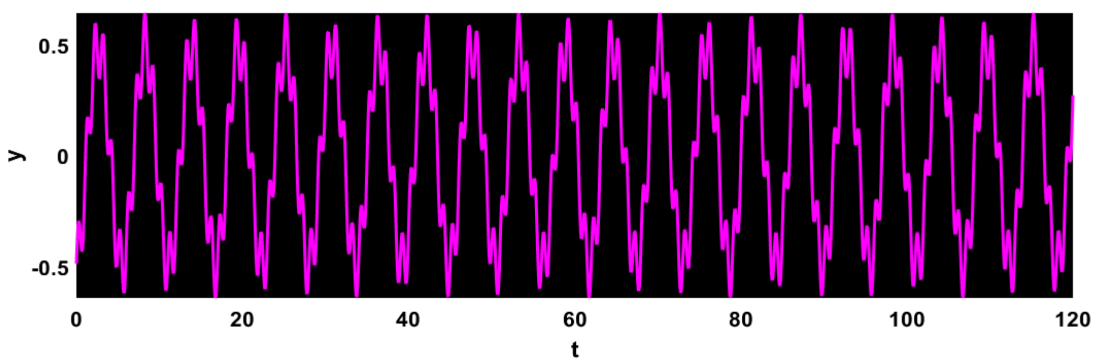
(a)



(b)



(c)



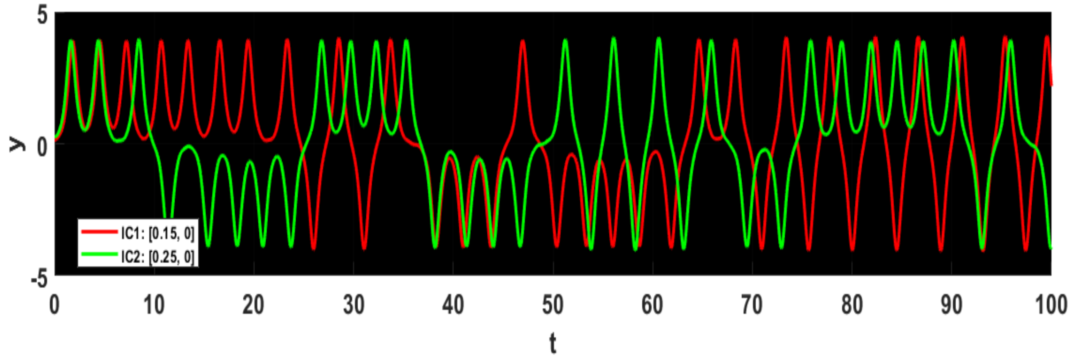
(d)

Figure 6

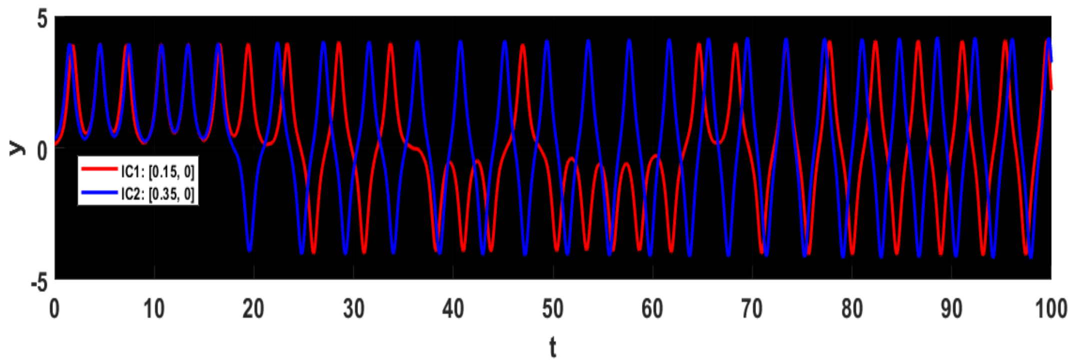
4.3 Sensitivity analysis

Sensitivity analysis is used to evaluate how altering input parameters impacts the results of a model, scheme, or process in a variety of fields, such as science, engineering, and economics. It comprises observing the changes in output that follow from methodically varying input values within a predefined range. Sensitivity analysis helps with risk assessment, improvement, and decision-making by identifying the input elements that have the most effects on results. It is crucial for understanding and managing uncertainty in intricate structures.

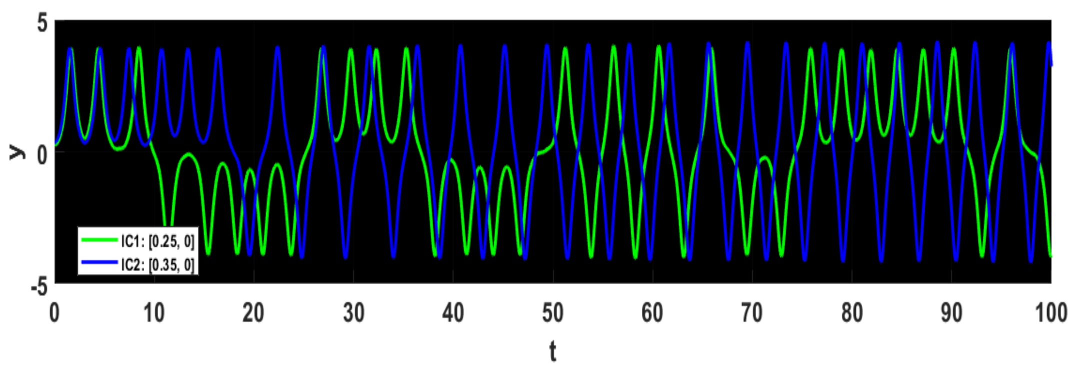
$$\begin{cases} \frac{df}{d\xi} & = v, \\ \frac{dv}{d\xi} & = -\sigma_1 v(\xi) - \sigma_2 (v(\xi))^3 \end{cases} \tag{21}$$



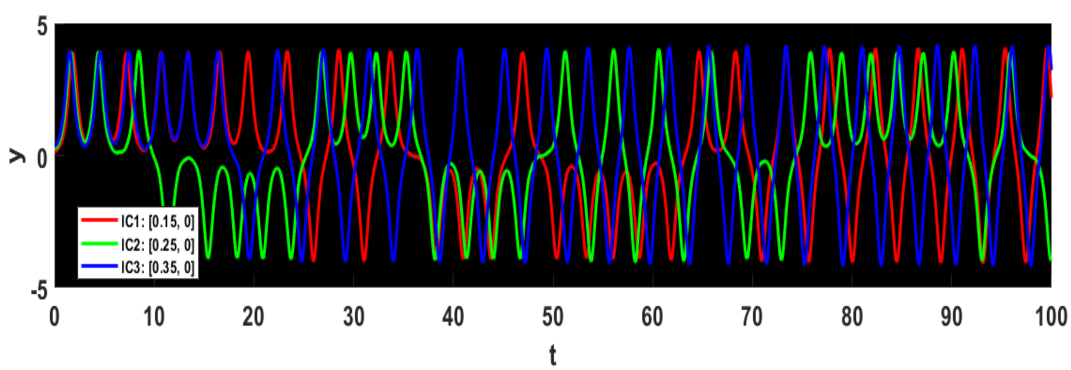
(a)



(b)



(c)



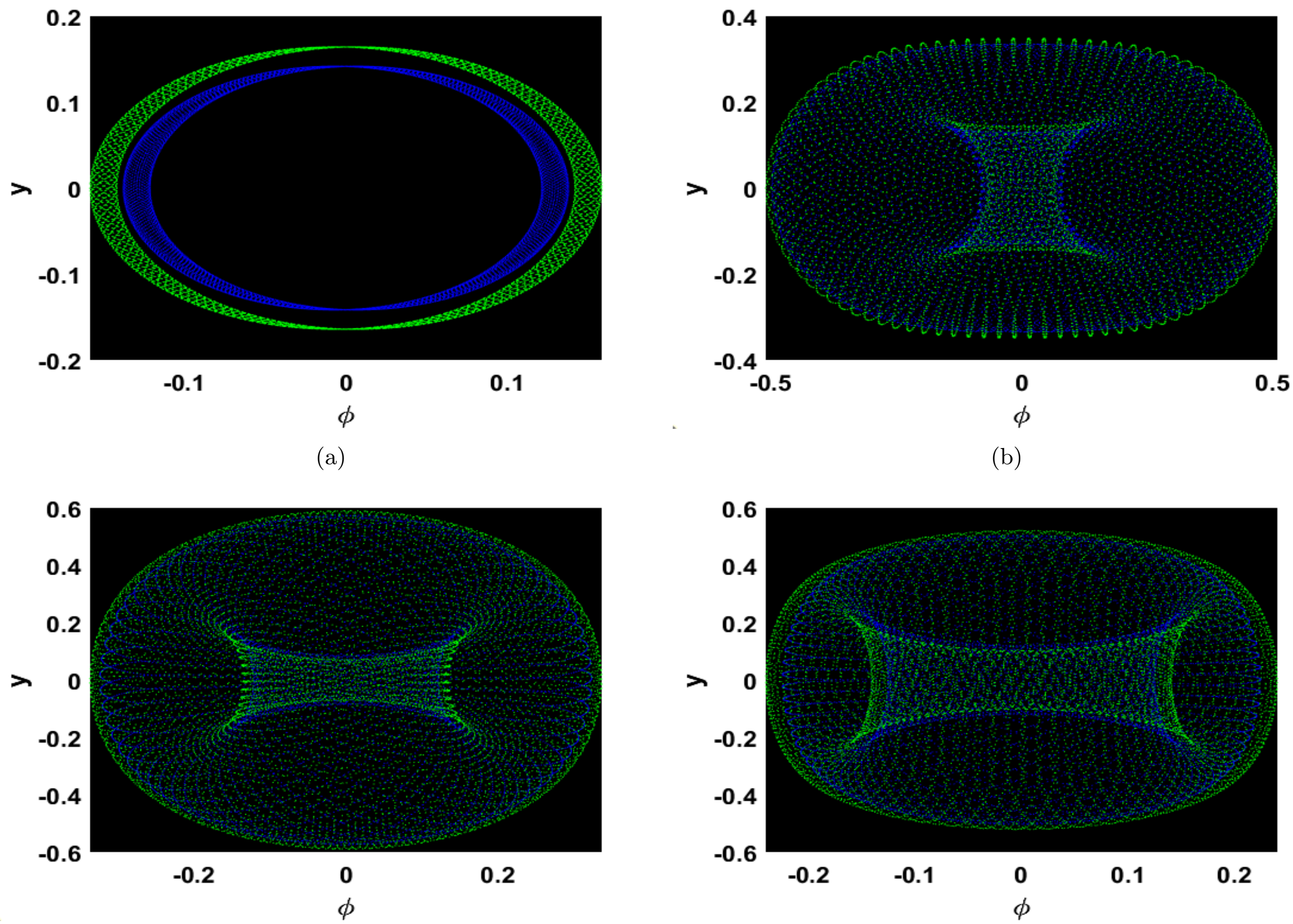
(d)

Figure 7

We investigated the proposed model's sensitivity by examining three different sets of initial conditions. The initial set is represented by the continuous red dashed curve and consists of the coordinates (f, v) = (0.1, 0). The second set, depicted by the blue curve, corresponds to the values $(f, v) = (0.4, 0)$; while the third set, shown by the dashed green curve, involves the values $(f, v) = (0.6, 0.6)$,Figure (16) presents two solutions: one for the initial condition $(\Omega, \xi) = (0.1, 0)$ in red dashed and the other for $(\Xi, \theta) = (0.4, 0)$ in blue. Figure (17) displays two solutions: $(f, v) = (0.1, 0)$ and $(f, v) = (0.6, 0.6)$. Figure (18) shows two solutions: $(f, v) = (0.4, 0)$ and $(f, v) = (0.6, 0.6)$. Figure (19) shows three solutions: (0.1, 0), (0.4, 0), and (0.6, 0.6). Slight alterations in the initial conditions can result in subtle shifts in the outcome of a dynamic system. Therefore, we conclude that the suggested system shows sensitivity, although not to an extreme level.

4.4 Poincare Map

The Poincaré map can be used to identify chaos in dynamical systems. This kind of return map documents the intersections of trajectories on a lower-dimensional cross-section. It essentially converts a continuous system into a discrete one by mapping each point to its subsequent intersection. A return map, which illustrates the history of a dynamical system across one or more periods without necessarily needing a cross-sectional plane, is a broader idea that maps a state of the system back to itself after some time. While quasi-periodic activity can be depicted as a closed curve or a collection of points in the Poincaré map, periodic phenomena can be categorized as a fixed point. The beginning of chaotic behavior is indicated by the existence of a distinct collection of points in the Poincaré map.



(a) (b) (c) (d)

Figure 8

4.5 Multistability

Understanding systems that can appear in multiple stable states depending on changing conditions requires multistability analysis. It makes it easier to predict when these states will change, providing deep insights into complex processes like stability and hysteresis. In fields including biology, engineering, chemical processes, economics, and finance, this analytical framework is essential for improving comprehension of system dynamics and equilibrium transitions. The analysis of the potential of multistability in the context of the system (16) will be expanded upon in this section. The system can disclose as many different kinds of perturbations that can impact an individual since it can reveal numerous dynamic types at once depending on changes in the initial conditions of the parameters. These consist of the periodic, quasi-periodic, and chaotic patterns, each of which manifests under particular circumstances.

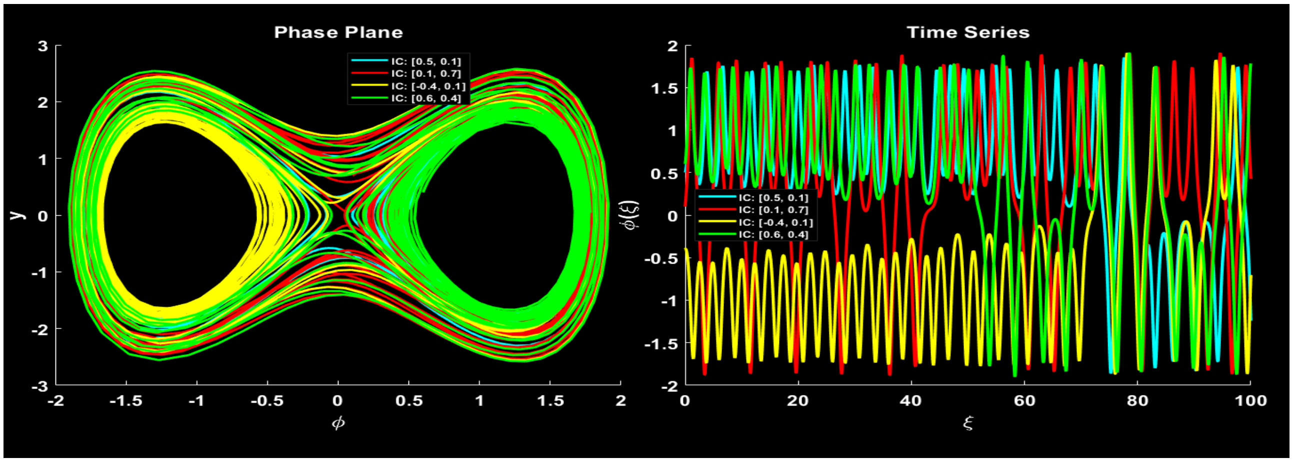


Figure 9

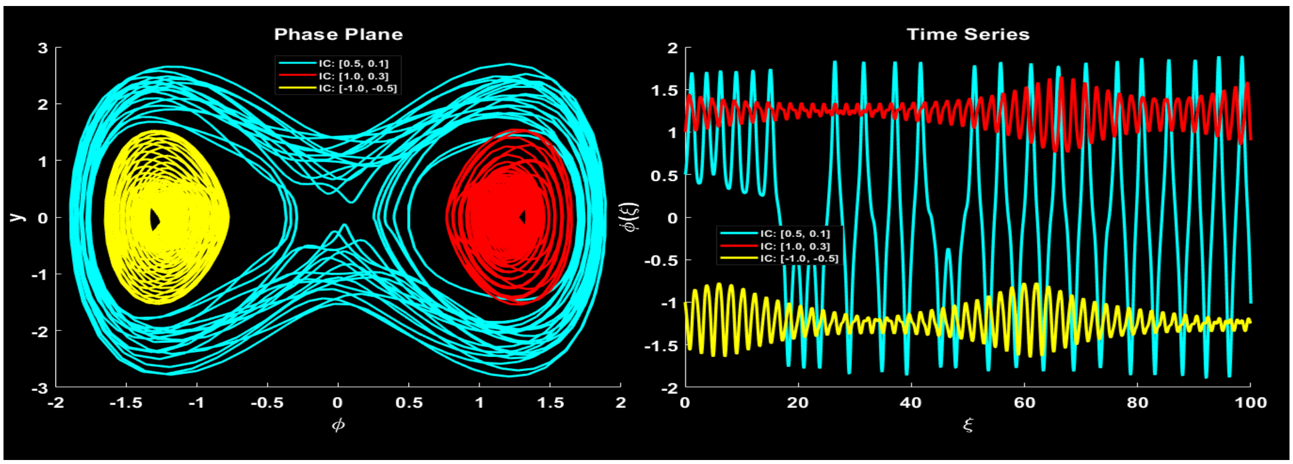


Figure 10

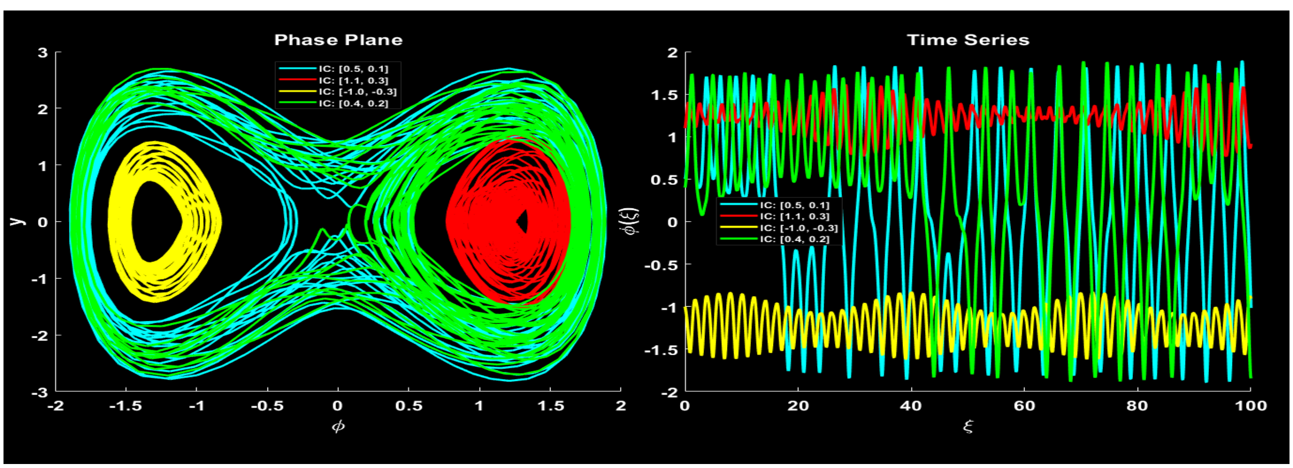


Figure 11

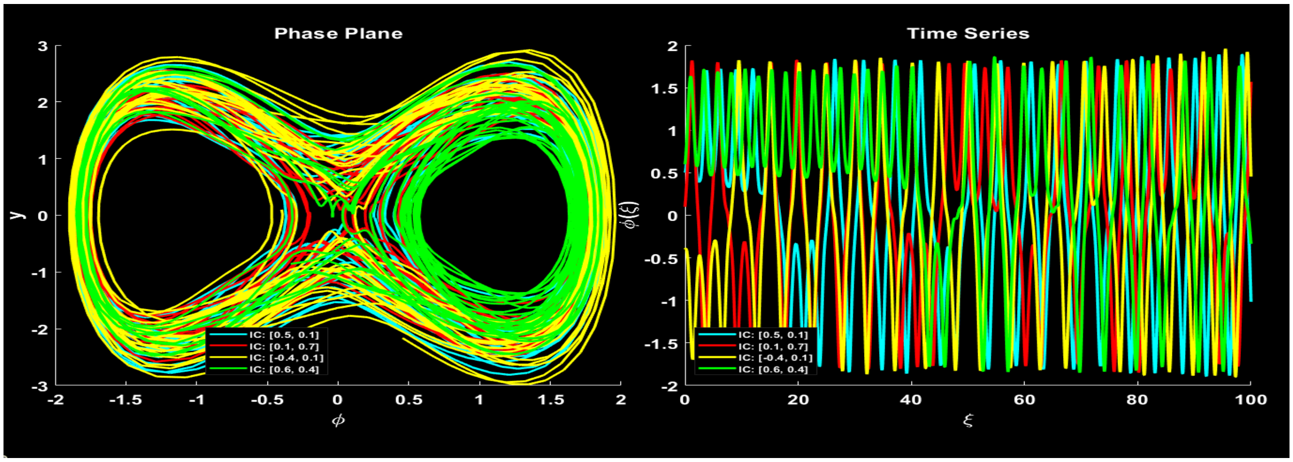


Figure 12

4.6 Lyapunov Exponent

The lyapunov exponent(LyE), which is based on chaos theory, a theory that studies dynamic systems and holds that a system is affected by its beginning conditions, is the method used to analyze stability. In particular, the LyE examines how neighboring trajectories differ from one another among data points at various time instances in order to analyze the local dynamic stability of a system, or the degree of sensitivity to minor perturbations that exist in a system. The dynamic stability of a system is indicated by the rate at which trajectories diverge and converge. A negative LyE number denotes trajectories that converge more over time and a locally stable system, while a positive LyE value denotes greater variance/divergence in trajectories, a more unstable system, and an inability to reduce local disturbances.

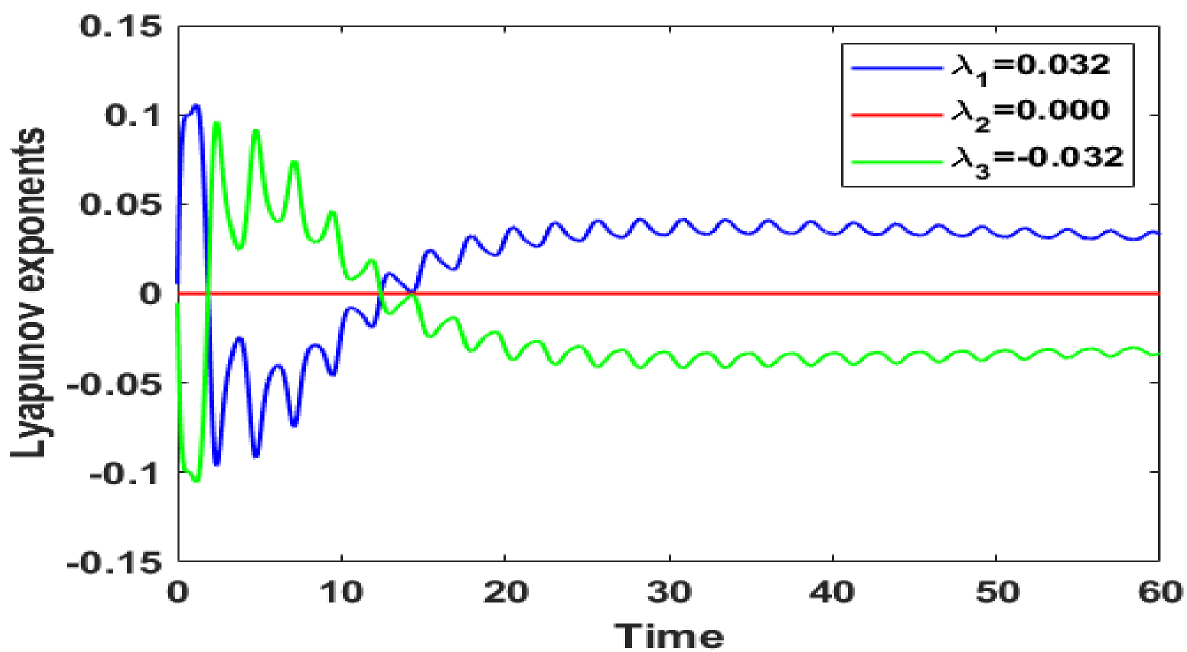


Figure 13

4.7 Bifurcation Analysis

Understanding how a dynamical system's qualitative behavior varies as a parameter is changed is the primary objective of bifurcation analysis. When this variance causes the system to undergo substantial changes and produce new dynamic behaviors, a bifurcation occurs. Equilibrium points, recurring patterns, or other system characteristics may develop, vanish, or become more or less stable as the parameter varies.

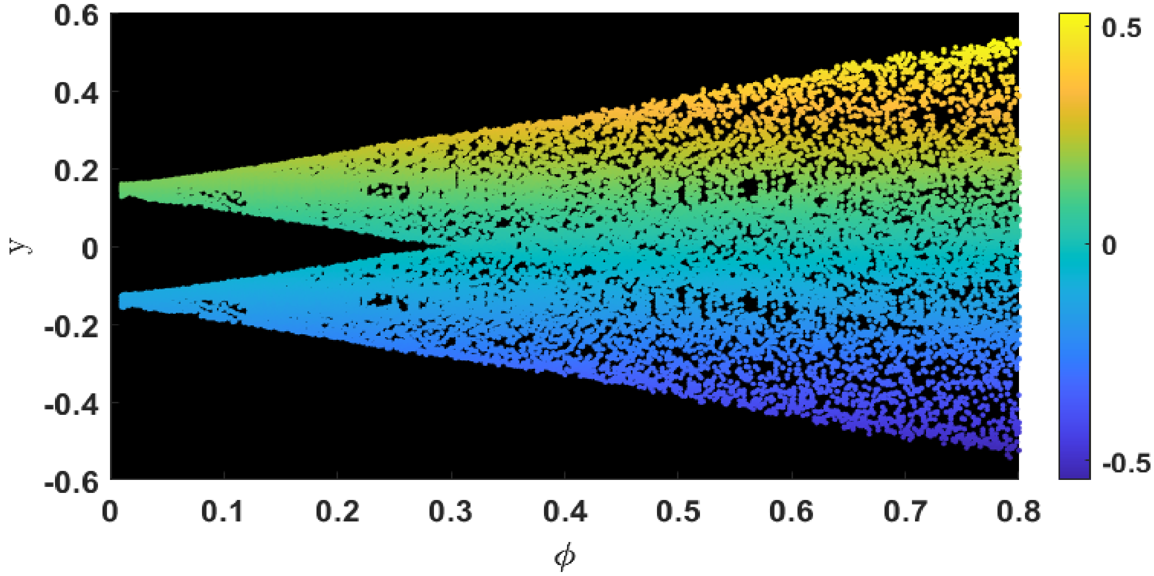


Figure 14

5 Conclusion

In this paper, we utilize the Sardar sub-equation and the modified Khater method to solve the new solitary wave equation. The obtained solutions manifest in various forms, including hyperbolic, trigonometric, and rational expressions. We systematically generated various soliton profiles to acquire ordinary differential equations, exploring a range of parameter values. To emphasize the physical significance of the proposed model, we produced 3D, 2D, density and contour graphs with the help of Mathematica and Matlab with carefully selected parameter values. These graphical illustrations serve to illustrate the outcomes of our study. Additionally, we explored the bifurcation analysis of the nonlinear Akbota equation using principles from bifurcation theory. Figure 14 and 15 show phase profiles under various parameter scenarios of bifurcation analysis. Furthermore, we investigate the sensitivity analysis of the Akbota equation.

References

[1] Hossain, M. M., & Sheikh, M. A. N. (2022). Bilinear form of the regularized long wave equation and its multi-soliton solutions. Partial Differential Equations in Applied Mathematics, 6, 100422.

[2] Ullah, M. S. (2023). Interaction solution to the (3+ 1)-D negative-order KdV first structure. Partial Differential Equations in Applied Mathematics, 8, 100566.

[3] Ullah, M. S., Roshid, H. O., & Ali, M. Z. (2023). New wave behaviors of the Fokas-Lenells model using three integration techniques. Plos one, 18(9), e0291071.

[4] Ullah, M. S., Roshid, H. O., & Ali, M. Z. (2024). New wave behaviors and stability analysis for the (2+ 1)-dimensional Zoomeron model. Optical and Quantum Electronics, 56(2), 240.

[5] He, J. H. (2005). Application of homotopy perturbation method to nonlinear wave equations. Chaos, Solitons & Fractals, 26(3), 695-700.

[6] Bekir, A. (2008). Application of the (G'/G)-expansion method for nonlinear evolution equations. Physics Letters A, 372(19), 3400-3406.

[7] Roshid, H. O., Akbar, M. A., Alam, M. N., Hoque, M. F., & Rahman, N. (2014). New extended (G'/G)-expansion method to solve nonlinear evolution equation: the (3+ 1)-dimensional potential-YTSF equation. SpringerPlus, 3(1), 122.

[8] Islam, Z., Sheikh, M. A. N., Roshid, H. O., Hossain, M. A., Taher, M. A., & Abdeljabbar, A. (2024). Stability and spin solitonic dynamics of the HFSC model: effects of neighboring interactions and crystal field anisotropy parameters. Optical and Quantum Electronics, 56(2), 190. [

[9] Rahman, M. A. (2014). The exp $(\phi(\eta)$-expansion method with application in the (1+ 1)-dimensional classical Boussinesq equations. Results in Physics, 4, 150-155.

[10] Roshid, M. M., Karim, M. F., Azad, A. K., Rahman, M. M., & Sultana, T. (2021). New solitonic and rogue wave solutions of a Klein–Gordon equation with quadratic nonlinearity. Partial Differential Equations in Applied Mathematics, 3, 100036.

[11] Sheikh, M. A. N., Taher, M. A., Hossain, M. M., & Akter, S. (2023). Variable coefficient exact solution of Sharma–Tasso–Olver model by enhanced modified simple equation method. Partial Differential Equations in Applied Mathematics, 7, 100527.

[12] Abdeljabbar, A. (2021). New double Wronskian exact solutions for a generalized (2+ 1)-dimensional nonlinear system with variable coefficients. Partial Differential Equations in Applied Mathematics, 3, 100022.

[13] Hossain, M. M., Roshid, M. M., Sheikh, M. A. N., Taher, M. A., & Roshid, H. O. (2022). Novel exact soliton solutions of Cahn–allen models with truncated M-fractional derivative. Int. J. Theor. Appl. Mech., 8(6), 112-120.

[14] Hossain, M. M., Abdeljabbar, A., Roshid, H. O., Roshid, M. M., & Sheikh, A. N. (2022). Abundant Bounded and Unbounded Solitary, Periodic, Rogue-Type Wave Solutions and Analysis of Parametric Effect on the Solutions to Nonlinear Klein–Gordon Model. Complexity, 2022(1), 8771583.

[15] El-Rashidy, K. (2020). New traveling wave solutions for the higher Sharma-Tasso-Olver equation by using extension exponential rational function method. Results in Physics, 17, 103066.

[16] Wang, K. J., Wang, G. D., & Shi, F. (2024). Diverse optical solitons to the Radhakrishnan–Kundu–Lakshmanan equation for the light pulses. Journal of Nonlinear Optical Physics & Materials, 33(06), 2350074.

[17] Yin, Y. H., Lü, X., & Ma, W. X. (2022). Bäcklund transformation, exact solutions and diverse interaction phenomena to a (3+ 1)-dimensional nonlinear evolution equation. Nonlinear Dynamics, 108(4), 4181-4194.

[18] Wang, K. J., Wang, G. D., & Shi, F. (2023). The pulse narrowing nonlinear transmission lines model within the local fractional calculus on the Cantor sets. COMPEL-The international journal for computation and mathematics in electrical and electronic engineering, 42(6), 1576-1593.

[19] Wang, K. J., & Xu, P. (2023). Generalized variational structure of the fractal modified KdV–Zakharov–Kuznetsov equation. Fractals, 31(07), 2350084.

[20] Wang, K. J., Xu, P., & Shi, F. (2023). Nonlinear dynamic behaviors of the fractional (3+ 1)-dimensional modified Zakharov–Kuznetsov equation. Fractals, 31(07), 2350088.

[21] Eroğlu, B. B. İ. (2023). Two-dimensional Cattaneo-Hristov heat diffusion in the half-plane. Mathematical Modelling and Numerical Simulation with Applications, 3(3), 281-296.

[22] Scott Russell, J. (1844). Report on Waves": Report of the fourteenth meeting of the British Association for the Advancement of Science. York, September.

[23] Horne, R. L. (2002). A (very) brief introduction to soliton theory in a class of nonlinear PDEs. Mathematical Sciences Proceedings, 3.

[24] Wadati, M. (2001). Introduction to solitons. Pramana, 57(5), 841-847.

[25] Temam, R. (2012). Infinite-dimensional dynamical systems in mechanics and physics (Vol. 68). Springer Science & Business Media.

[26] Arnold, L. (2006). Random dynamical systems. In Dynamical Systems: Lectures Given at the 2nd Session of the Centro Internazionale Matematico Estivo (CIME) held in Montecatini Terme, Italy, June 13–22, 1994 (pp. 1-43). Berlin, Heidelberg: Springer Berlin Heidelberg.

[27] Bécus, G. A. (1977). Random generalized solutions to the heat equation. Journal of Mathematical Analysis and Applications, 60(1), 93-102.

[28] Blömker, D., & Mohammed, W. W. (2013). Amplitude equations for SPDEs with cubic nonlinearities. Stochastics An International Journal of Probability and Stochastic Processes, 85(2), 181-215.

[29] Jetschke, G., & Manthey, R. (1986). On a linear reaction diffusion equation with white noise on the whole real line. Friedrich-Schiller-Univ.

[30] Zhu, S. D. (2008). The generalizing Riccati equation mapping method in non-linear evolution equation: application to (2+ 1)-dimensional Boiti–Leon–Pempinelle equation. Chaos, Solitons & Fractals, 37(5), 1335-1342.

[31] Zhang, Z., Xia, F. L., Li, X. P. (2013). Bifurcation analysis and the travelling wave solutions of the Klein–Gordon–Zakharov equations. Pramana, 80(1), 41-59.

[32] Roshid, M. M., Rahman, M. M. (2024). Bifurcation analysis, modulation instability and optical soliton solutions and their wave propagation insights to the variable coefficient nonlinear Schrödinger equation with Kerr law nonlinearity. Nonlinear Dynamics, 112(18), 16355-16377.

[33] Dijkstra, H. A., Wubs, F. W. (2023). Bifurcation Analysis of Fluid Flows. Cambridge University Press.

[34] Iqbal, I., Boulaaras, S. M., Althobaiti, S., Althobaiti, A., Rehman, H. U. (2025). Exploring soliton dynamics in the nonlinear Helmholtz equation: bifurcation, chaotic behavior, multistability, and sensitivity analysis. Nonlinear Dynamics, 113(13), 16933-16954.

[35] Khalifa, A., Ahmed, H., Ahmed, K. K. (2024). Construction of exact solutions for a higher-order stochastic modified Gerdjikov–Ivanov model using the Imetf method. Physica Scripta.

[36] Deeks, J. J., Higgins, J. P., Altman, D. G.,Cochrane Statistical Methods Group. (2019). Analysing data and undertaking meta-analyses. Cochrane handbook for systematic reviews of interventions, 241-284.

[37] Schneeweiss, S. (2006). Sensitivity analysis and external adjustment for unmeasured confounders in epidemiologic database studies of therapeutics. Pharmacoepidemiology and drug safety, 15(5), 291-303.